

# REAL EIGENVALUES OF A NON-SELF-ADJOINT PERTURBATION OF THE SELF-ADJOINT ZAKHAROV-SHABAT OPERATOR

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**ABSTRACT.** We study the eigenvalues of the self-adjoint Zakharov-Shabat operator corresponding to the defocusing nonlinear Schrödinger equation in the inverse scattering method. Real eigenvalues exist when the square of the potential has a simple well. We derive two types of quantization condition for the eigenvalues by using the exact WKB method, and show that the eigenvalues stay real for a sufficiently small non-self-adjoint perturbation when the potential has some  $\mathcal{PT}$ -like symmetry.

## 1. INTRODUCTION

We consider the eigenvalue problem

$$(1.1) \quad L\mathbf{u}(x) = \lambda\mathbf{u}(x),$$

for the first order  $2 \times 2$  differential system on the line:

$$L := \begin{pmatrix} ih \frac{d}{dx} & -iA(x) \\ iA(x) & -ih \frac{d}{dx} \end{pmatrix},$$

where  $h$  is a small positive parameter,  $\lambda$  is a spectral parameter,  $\mathbf{u}(x)$  is a column vector, and  $A(x)$  is a real-valued potential. This operator is called the Zakharov-Shabat operator, which is one of the two operators in the Lax pair for the defocusing nonlinear Schrödinger equation:

$$ih \frac{\partial \psi}{\partial t} + \frac{h^2}{2} \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2 \psi = 0, \quad \psi = \psi(t, x),$$

and the scattering theory of  $L$  plays an important role in the analysis of the solutions of the initial value problem for this equation.

The operator  $L$  is self-adjoint, and it is expected that  $L$  has real eigenvalues when  $A(x)^2$  has a well. In the first part of our study, we derive the Bohr-Sommerfeld type quantization condition for the eigenvalues of  $L$  under the following assumption.

**Assumption (A1).** Let  $A(x)$  be a real-valued function analytic in  $D := \{z \in \mathbb{C}; |\operatorname{Im} z| < \delta\}$  for some  $\delta > 0$ , and  $\lambda_0$  a positive real number satisfying the following conditions:

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*Keywords:* Zakharov-Shabat eigenvalue problem, exact WKB method, quantization condition.

- (1) There exist two real numbers,  $\alpha_0$  and  $\beta_0$  ( $\alpha_0 < \beta_0$ ) such that  $|A(x)| = \lambda_0, x \in \mathbb{R}$  if and only if  $x = \alpha_0, \beta_0$ .
- (2)  $A'(\alpha_0)A'(\beta_0) \neq 0$ .
- (3)  $|A(x)| < \lambda_0$  for  $\alpha_0 < x < \beta_0$ , and  $|A(x)| > \lambda_0$  for  $x < \alpha_0$  and  $x > \beta_0$ .
- (4)  $\liminf_{|x| \rightarrow \infty} |A(x)| > \lambda_0$ .

This assumption permits two types of potentials. One is a simple well type where  $A(\alpha_0) = A(\beta_0)$ , and the other is monotonic type where  $A(\alpha_0) = -A(\beta_0)$ . In both cases,  $A(x)^2$  has a simple well, see Figure 1.

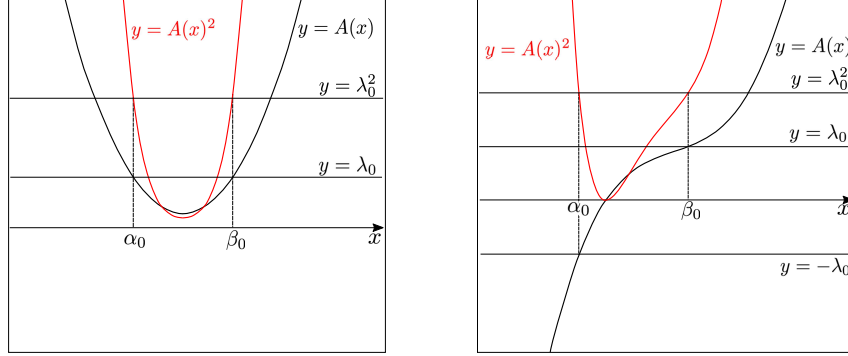


FIGURE 1. Examples of the potential  $A(x)$ .

For  $\lambda \in \mathbb{R}$  close enough to  $\lambda_0$ , the function  $\lambda^2 - A(x)^2$  has exactly two real zeros  $\alpha(\lambda)$  and  $\beta(\lambda)$ , close to  $\alpha_0$  and  $\beta_0$  respectively, and we define the action integral

$$(1.2) \quad I(\lambda) := \int_{\alpha(\lambda)}^{\beta(\lambda)} \sqrt{\lambda^2 - A(t)^2} dt.$$

Then, we obtain the following quantization conditions.

**Theorem 1.1.** *Assume (A1). In the case  $A(\alpha_0) = A(\beta_0)$ , there exist positive constants  $\delta$  and  $h_0$ , and a function  $r_+(\lambda, h)$  bounded on  $[\lambda_0 - \delta, \lambda_0 + \delta] \times (0, h_0]$  such that  $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$  is an eigenvalue of  $L$  for  $h \in (0, h_0]$  if and only if*

$$(1.3) \quad I(\lambda) = \left(k + \frac{1}{2}\right) \pi h + h^2 r_+(\lambda, h)$$

*holds for some integer  $k$ . In the case  $A(\alpha_0) = -A(\beta_0)$ , there exist positive constants  $\delta$  and  $h_0$ , and a function  $r_-(\lambda, h)$  bounded on  $[\lambda_0 - \delta, \lambda_0 + \delta] \times (0, h_0]$  such that  $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$  is an eigenvalue of  $L$  for  $h \in (0, h_0]$  if and only if*

$$(1.4) \quad I(\lambda) = k\pi h + h^2 r_-(\lambda, h)$$

*holds for some integer  $k$ .*

Next, we add a small complex perturbation to the potential  $A(x)$ :

$$A_\varepsilon(x) = A(x) + i\varepsilon B(x)$$

with a real-valued function  $B(x)$  and a positive small parameter  $\varepsilon$ , and consider the eigenvalues of  $L_\varepsilon$

$$L_\varepsilon := \begin{pmatrix} ih \frac{d}{dx} & -iA_\varepsilon(x) \\ iA_\varepsilon(x) & -ih \frac{d}{dx} \end{pmatrix}.$$

This operator is no longer self-adjoint, and eigenvalues become complex in general.

In the case of Schrödinger operator,  $\mathcal{PT}$ -symmetry has been expected to be an alternative to the self-adjointness in order to have real eigenvalues. In recent studies, Boussekkine and Mecherout considered in for the Schrödinger operator with  $\mathcal{PT}$ -symmetry

$$P_\varepsilon := -h^2 \frac{d^2}{dx^2} + V(x) + i\varepsilon W(x),$$

where  $V(x)$  is a simple well even function and  $W(x)$  is an odd function, and showed that reality of eigenvalues also holds for sufficiently small  $\varepsilon$  and  $h$ . After that, Boussekkine, Mecherout, Ramond and Sjöstrand studied in [8] the double well case with  $\mathcal{PT}$ -symmetry, and found that the eigenvalues stay real only for exponentially small  $\varepsilon$  with respect to  $h$ .

In this paper, we continue in this direction and prove that a sufficiently small complex perturbation of the self-adjoint Zakharov-Shabat operator  $L_\varepsilon$  has real eigenvalues when  $A(x)$  and  $B(x)$  have some  $\mathcal{PT}$ -like symmetry in the case where  $A(x)^2$  has a simple well, even though the perturbed operator  $L_\varepsilon$  is non-self-adjoint. Recalling that the condition where  $P_\varepsilon$  is  $\mathcal{PT}$ -symmetric is equivalent to one where  $V(x)$  is an even function and  $W(x)$  is an odd function (see [4] or [8]), we assume the following symmetry properties for  $A(x)$  and  $B(x)$ .

**Assumption (A2).** Let  $B(x)$  be real-valued, analytic and bounded on  $\mathbb{R}$ .  $A(x)$  and  $B(x)$  satisfy for  $x \in \mathbb{R}$  either

$$(1.5) \quad A(x) = A(-x), \quad B(x) = -B(-x),$$

or

$$(1.6) \quad A(x) = -A(-x), \quad B(x) = B(-x).$$

The following theorem shows that the eigenvalues of  $L_\varepsilon$  are real for sufficiently small  $\varepsilon$  and  $h$ .

**Theorem 1.2.** *Assume (A1) and (A2). Then there exist positive constants  $\varepsilon_0$  and  $h_0$  such that  $\sigma(L_\varepsilon) \cap \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon_0\} \subset \mathbb{R}$  when  $0 < \varepsilon \leq \varepsilon_0$  and  $0 < h \leq h_0$ .*

To prove Theorem 1.1 and 1.2, we use the exact WKB method. In Section 2, we mention the exact WKB solutions for (1.1) and introduce three important properties. These exact WKB solutions are used in Section 3 to derive the quantization conditions (1.3) and (1.4). After that, we consider the perturbed case, and give the proof for Theorem 1.2 in Section 4.

## 2. EXACT WKB SOLUTIONS

We construct solutions to (1.1) by the exact WKB method. This method was proposed by Gérard and Grigis in [7], and extended to  $2 \times 2$  systems by Fujiié, Lasser and Nédélec in [5].

Before the construction, we assume that  $\Omega$  is a simply connected open subset of  $D$ , where  $A(x)^2 - \lambda^2$  does not vanish. Following [5], we can construct exact WKB solutions for (1.1) in the form

$$(2.1) \quad \mathbf{u}^\pm(x, h; \gamma, x_0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} e^{\pm z(x; \gamma)/h} Q(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\frac{1 \pm 1}{2}} \mathbf{w}^\pm(x, h; x_0)$$

with base points  $\gamma \in D$  and  $x_0 \in \Omega$ . Here,  $z(x; \gamma)$  is a phase function

$$z(x; \gamma) := \int_\gamma^x \sqrt{A(t)^2 - \lambda^2} dt,$$

$Q(x)$  is a  $2 \times 2$  matrix function

$$(2.2) \quad Q(x) = \begin{pmatrix} H(x)^{-1} & H(x)^{-1} \\ iH(x) & -iH(x) \end{pmatrix}, \quad H(x) = \left( \frac{A(x) + \lambda}{A(x) - \lambda} \right)^{\frac{1}{4}},$$

and  $\mathbf{w}^\pm(x, h; x_0)$  are the series

$$\mathbf{w}^\pm(x, h; x_0) = \begin{pmatrix} w_{\text{even}}^\pm(x, h; x_0) \\ w_{\text{odd}}^\pm(x, h; x_0) \end{pmatrix} := \sum_{n=0}^{\infty} \begin{pmatrix} w_{2n}^\pm(x, h) \\ w_{2n-1}^\pm(x, h) \end{pmatrix}$$

constructed by the recurrence equations

$$(2.3) \quad \begin{cases} w_{-1}^\pm = 0, \quad w_0^\pm = 1, \\ \frac{d}{dx} w_{2n}^\pm = c(x) w_{2n-1}^\pm, \quad c(x) = \frac{H'(x)}{H(x)}, \\ \left( \frac{d}{dx} \pm \frac{2z'}{h} \right) w_{2n-1}^\pm = c(x) w_{2n-2}^\pm, \end{cases}$$

and the initial conditions

$$(2.4) \quad w_n^\pm|_{x=x_0} = 0 \quad (n \geq 1).$$

These solutions constructed above formally satisfy (1.1). We recall here the following three propositions. The proofs are found in [5] or [7]. The first is about the convergence of series.

**Proposition 2.1.** *Two series  $w_{\text{even}}^\pm(x, h; x_0)$  and  $w_{\text{odd}}^\pm(x, h; x_0)$  are absolutely convergent in a neighborhood of  $x_0$ . Furthermore,  $w_{\text{even}}^\pm(x, h; x_0)$  and  $w_{\text{odd}}^\pm(x, h; x_0)$  are analytic functions in  $\Omega$ .*

The second property is about the Wronskian for two different types of exact WKB solutions.

**Proposition 2.2.** *Let  $\gamma, x_0, x_1 \in \Omega$  be the base points. Then, the exact WKB solutions  $\mathbf{u}^\pm(x, h; \gamma, x_0)$  and  $\mathbf{u}^\pm(x, h; \gamma, x_1)$  satisfy*

$$\mathcal{W}(\mathbf{u}^\pm(x, h; \gamma, x_0), \mathbf{u}^\mp(x, h; \gamma, x_1)) = \mp 4w_{\text{even}}^\pm(x_1, h; x_0),$$

where  $\mathcal{W}(\mathbf{f}, \mathbf{g}) := \det(\mathbf{f}, \mathbf{g})$ . This is called the Wronskian formula.

The final proposition is about the asymptotic property of the exact WKB solution. Let  $x_0 \in \Omega$  be fixed.

**Definition 2.3.** We denote by  $\Omega_{\pm}$  the subset of all  $x \in \Omega$  such that there exists a path in  $\Omega$  from  $x_0$  to  $x$  along which  $\pm \operatorname{Re} z(x; x_0)$  is strictly increasing.

**Theorem 2.4.** The functions  $w_{\text{even}}^{\pm}(x, h; x_0)$  and  $w_{\text{odd}}^{\pm}(x, h; x_0)$  have the asymptotic expansions as  $h \rightarrow 0$ :

$$w_{\text{even}}^{\pm}(x, h; x_0) - \sum_{n=0}^N w_{2n}^{\pm}(x, h; x_0) = \mathcal{O}(h^{N+1}),$$

$$w_{\text{odd}}^{\pm}(x, h; x_0) - \sum_{n=0}^N w_{2n+1}^{\pm}(x, h; x_0) = \mathcal{O}(h^{N+1}),$$

in all compact subsets of  $\Omega_{\pm}$ .

To find the domain  $\Omega_{\pm}$ , we usually consider the Stokes lines, which are level curves of the real part of  $z(x; \gamma)$ . In particular, the Stokes lines passing through the point  $\gamma_0 \in D$  are defined as the set

$$\left\{ x \in D; \operatorname{Re} z(x; \gamma_0) = \operatorname{Re} \int_{\gamma_0}^x \sqrt{A(t)^2 - \lambda^2} dt = 0 \right\}.$$

Along a path which intersects transversally with the Stokes lines,  $\operatorname{Re} z(x)$  or  $-\operatorname{Re} z(x)$  is strictly increasing.

### 3. QUANTIZATION CONDITION FOR THE EIGENVALUES OF $L$

Here we find the quantization condition under Assumption (A1). This is derived from the connection problem of the solutions near the points  $\alpha(\lambda)$  and  $\beta(\lambda)$  which are zeros of  $A(x)^2 - \lambda^2$ .

Now, we choose  $\alpha(\lambda)$  and  $\beta(\lambda)$  for base points of the phase function  $z(x)$ , and consider the Stokes lines which pass through  $\alpha(\lambda)$  and  $\beta(\lambda)$ . By a simple calculation, we see that the Stokes lines emanate from  $\alpha$  at angles of  $0, 2\pi/3$  and  $4\pi/3$ , and emanate from  $\beta$  at angles of  $\pi/3, \pi$  and  $5\pi/3$ .

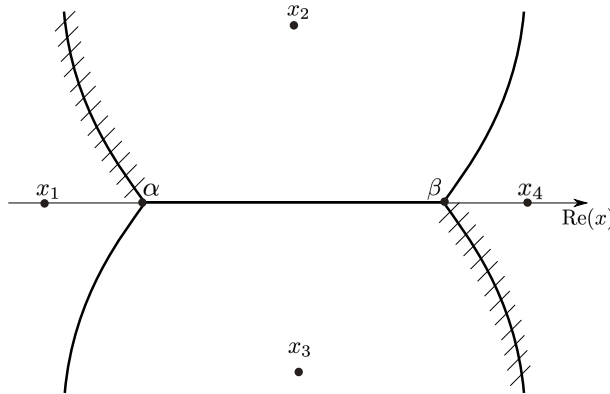


FIGURE 2. The Stokes lines and base points

Those Stokes lines separate the complex plane into four sectors as in Figure 2. As  $\sqrt{A(x)^2 - \lambda^2}$  and  $H(x)$  are multi-valued functions on the complex plane with singularities at  $\alpha$  and  $\beta$ , we set branch cuts emanating at an angle of  $2\pi/3$  from  $\alpha$  and an angle of  $5\pi/3$  from  $\beta$  respectively. We choose the branches such that  $\operatorname{Re}\sqrt{A(x)^2 - \lambda^2}$  and  $H(x)$  are positive on a part of the real axis  $\operatorname{Re}(x) > \beta$ .

We take base points for  $\mathbf{w}^\pm(x, h)$  in each sector as in Figure 2, and define the exact WKB solutions:

$$(3.1) \quad \begin{cases} \mathbf{u}_1 = \mathbf{u}^+(x, h; \alpha, x_1) \\ \mathbf{u}_2 = \mathbf{u}^+(x, h; \alpha, x_2), \quad \tilde{\mathbf{u}}_2 = \mathbf{u}^+(x, h; \beta, x_2), \\ \mathbf{u}_3 = \mathbf{u}^-(x, h; \alpha, x_3), \quad \tilde{\mathbf{u}}_3 = \mathbf{u}^-(x, h; \beta, x_3), \\ \mathbf{u}_4 = \mathbf{u}^-(x, h; \beta, x_4). \end{cases}$$

Then, we represent  $\mathbf{u}_1$  as a linear combination of  $\mathbf{u}_2$  and  $\mathbf{u}_3$ :

$$(3.2) \quad \mathbf{u}_1 = c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3,$$

and  $\mathbf{u}_4$  as

$$(3.3) \quad \mathbf{u}_4 = \tilde{c}_2 \tilde{\mathbf{u}}_2 + \tilde{c}_3 \tilde{\mathbf{u}}_3,$$

where each coefficient depends on  $h$  and  $\lambda$ . We calculate those coefficients by using Theorems 2.2 and 2.4, and obtain the following.

**Lemma 3.1.** *Assume (A1). In the two cases  $A(\alpha) = \pm A(\beta)$ , the connection coefficients  $c_i$  and  $\tilde{c}_j$  ( $i, j \in \{2, 3\}$ ) satisfy*

$$c_2 \tilde{c}_3 = 1 + \mathcal{O}(h), \quad \tilde{c}_2 c_3 = \mp 1 + \mathcal{O}(h).$$

as  $h \rightarrow 0$ .

*Proof.* Each coefficient is represented in terms of the Wronskians as

$$\begin{aligned} c_2 &= \frac{\mathcal{W}(\mathbf{u}_1, \mathbf{u}_3)}{\mathcal{W}(\mathbf{u}_2, \mathbf{u}_3)}, \quad c_3 = \frac{\mathcal{W}(\mathbf{u}_1, \mathbf{u}_2)}{\mathcal{W}(\mathbf{u}_3, \mathbf{u}_2)}, \\ \tilde{c}_2 &= \frac{\mathcal{W}(\mathbf{u}_4, \tilde{\mathbf{u}}_3)}{\mathcal{W}(\tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3)}, \quad \tilde{c}_3 = \frac{\mathcal{W}(\mathbf{u}_4, \tilde{\mathbf{u}}_2)}{\mathcal{W}(\tilde{\mathbf{u}}_3, \tilde{\mathbf{u}}_2)}. \end{aligned}$$

For  $c_2$  and  $\tilde{c}_3$ , we see that

$$(3.4) \quad c_2 = \frac{w_{\text{even}}^+(x_3, h; x_1)}{w_{\text{even}}^+(x_3, h; x_2)}, \quad \tilde{c}_3 = \frac{w_{\text{even}}^+(x_4, h; x_2)}{w_{\text{even}}^+(x_3, h; x_2)}.$$

by Theorem 2.2.

Let  $\Gamma(x_i, x_j)$  denote a path from  $x_i$  to  $x_j$ . We take  $\Gamma(x_1, x_3)$ ,  $\Gamma(x_2, x_3)$  and  $\Gamma(x_2, x_4)$ , and then notice that they intersect the Stokes lines, see Figure 3. Moreover,  $\operatorname{Re} z(x, \cdot)$  increases as  $\operatorname{Re}(x)$  increases, or  $\operatorname{Im}(x)$  decreases. Therefore,  $\operatorname{Re} z(x, \cdot)$  is strictly increasing along those paths.

According to Theorem 2.4, we obtain

$$\begin{aligned} w_{\text{even}}^+(x_3, h; x_1) &= 1 + \mathcal{O}(h), \quad w_{\text{even}}^+(x_3, h; x_2) = 1 + \mathcal{O}(h), \\ w_{\text{even}}^+(x_4, h; x_2) &= 1 + \mathcal{O}(h) \end{aligned}$$

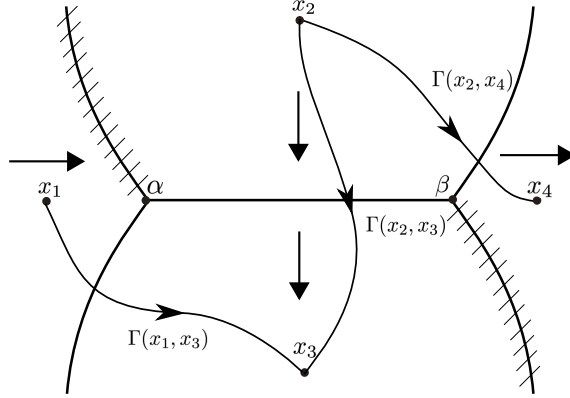


FIGURE 3. Examples of  $\Gamma(x_i, x_j)$ . Arrows indicate directions along which  $\text{Re} z(x)$  increases.

as  $h \rightarrow 0$ , and we see that

$$c_2 \tilde{c}_3 = \frac{1 + \mathcal{O}(h)}{1 + \mathcal{O}(h)} \frac{1 + \mathcal{O}(h)}{1 + \mathcal{O}(h)} = 1 + \mathcal{O}(h) \quad (h \rightarrow 0)$$

from (3.4). This holds in both cases  $A(\alpha) = \pm A(\beta)$ .

To calculate the Wronskian  $\mathcal{W}(\mathbf{u}_1, \mathbf{u}_2)$ , we recall that there exists a branch cut between  $x_1$  and  $x_2$ . For this, we have to represent  $\mathbf{u}_1$  or  $\mathbf{u}_2$  by different branches. Let  $\hat{x}$  denote a point obtained by rotating  $x$  by the angle of  $-2\pi$  around  $\alpha$ , that is,

$$\hat{x} - \alpha = e^{-2\pi i}(x - \alpha).$$

Then, we rewrite  $\mathbf{u}_2$  in terms of  $\hat{x}$ . When  $A(\alpha) = \lambda$ ,

$$\sqrt{A(x) - \lambda} = \sqrt{e^{2\pi i}(A(\hat{x}) - \lambda)} = -\sqrt{A(\hat{x}) - \lambda}.$$

On the other hand, when  $A(\alpha) = -\lambda$ ,

$$\sqrt{A(x) + \lambda} = \sqrt{e^{2\pi i}(A(\hat{x}) + \lambda)} = -\sqrt{A(\hat{x}) + \lambda}.$$

Therefore, there is a sign change

$$(3.5) \quad +z(x; \alpha) = -z(\hat{x}; \alpha)$$

in both cases  $A(\alpha) = \pm\lambda$ . Since the sign of  $z(x)$  changes and  $c(x) = c(\hat{x})$ , we find from the recurrence equation (2.3) that

$$(3.6) \quad \mathbf{w}^+(x, h; x_2) = \mathbf{w}^-(\hat{x}, h; \hat{x}_2).$$

The representation of the function  $H(x)$  is different in the cases where  $A(\alpha) = \lambda$  or  $A(\alpha) = -\lambda$ . When  $A(\alpha) = \lambda$ ,

$$H(x) = \left( \frac{A(x) + \lambda}{A(x) - \lambda} \right)^{\frac{1}{4}} = \left( \frac{1}{e^{2\pi i}} \frac{A(\hat{x}) + \lambda}{A(\hat{x}) - \lambda} \right)^{\frac{1}{4}} = e^{-\frac{\pi i}{2}} \left( \frac{A(\hat{x}) + \lambda}{A(\hat{x}) - \lambda} \right)^{\frac{1}{4}}.$$

By contrast, when  $A(\alpha) = -\lambda$ ,

$$H(x) = \left( \frac{A(x) + \lambda}{A(x) - \lambda} \right)^{\frac{1}{4}} = \left( e^{2\pi i} \frac{A(\hat{x}) + \lambda}{A(\hat{x}) - \lambda} \right)^{\frac{1}{4}} = e^{\frac{\pi i}{2}} \left( \frac{A(\hat{x}) + \lambda}{A(\hat{x}) - \lambda} \right)^{\frac{1}{4}}.$$

That is,  $H(x) = \mp i H(\hat{x})$  holds with  $A(\alpha) = \pm \lambda$ . In addition, this leads to

$$(3.7) \quad Q(x) = \pm i Q(\hat{x}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From (3.5), (3.6), and (3.7), we can rewrite  $\mathbf{u}_2$  as

$$\mathbf{u}_2 = \pm i \mathbf{u}^-(\hat{x}, h; \alpha, \hat{x}_2),$$

and obtain

$$\begin{aligned} \mathcal{W}(\mathbf{u}_1, \mathbf{u}_2) &= \mathcal{W}(\mathbf{u}^+(x, h; \alpha, x_1), \pm i \mathbf{u}^-(\hat{x}, h; \alpha, \hat{x}_2)) \\ &= \mp 4i \det Q w_{even}^+(\hat{x}_2, h; x_1). \end{aligned}$$

in each case  $A(\alpha) = \pm \lambda$ .

In the same way, we represent  $\widetilde{\mathbf{u}}_3$  by the other branch to calculate  $\mathcal{W}(\mathbf{u}_4, \widetilde{\mathbf{u}}_3)$ . Let  $\widetilde{x}$  denote a point obtained by rotating  $x$  by the angle of  $-2\pi$  around  $\beta$ . When  $A(\beta) = \pm \lambda$ ,  $\widetilde{\mathbf{u}}_3$  is rewritten as

$$\widetilde{\mathbf{u}}_3 = \pm i \mathbf{u}^+(\widetilde{x}, h; \beta, \widetilde{x}_3).$$

Therefore,  $\mathcal{W}(\mathbf{u}_4, \widetilde{\mathbf{u}}_3)$  is calculated as

$$\begin{aligned} \mathcal{W}(\mathbf{u}_4, \widetilde{\mathbf{u}}_3) &= -\mathcal{W}(\pm i \mathbf{u}^+(x, h; \beta, \widetilde{x}_3), \mathbf{u}^-(x, h; \beta, x_4)) \\ &= \pm 4i \det Q w_{even}^+(x_4, h; \widetilde{x}_3). \end{aligned}$$

We can find  $\Gamma(x_1, \hat{x}_2)$  and  $\Gamma(\widetilde{x}_3, x_4)$  along which  $\text{Re} z(x, \cdot)$  strictly increasing, and obtain

$$w_{even}^+(\hat{x}_2, h; x_1) = 1 + \mathcal{O}(h), \quad w_{even}^+(x_4, h; \widetilde{x}_3) = 1 + \mathcal{O}(h)$$

as  $h \rightarrow 0$ .

As a result, we obtain that when  $A(\alpha)A(\beta) > 0$ ,

$$c_3 \widetilde{c}_2 = i^2 \frac{w_{even}^+(\hat{x}_2, h; x_1)}{w_{even}^+(x_3, h; x_2)} \frac{w_{even}^+(x_4, h; \widetilde{x}_3)}{w_{even}^+(x_3, h; x_2)} = -1 + \mathcal{O}(h)$$

as  $h \rightarrow 0$ . In the case  $A(\alpha)A(\beta) < 0$ ,

$$c_3 \widetilde{c}_2 = -i^2 \frac{w_{even}^+(\hat{x}_2, h; x_1)}{w_{even}^+(x_3, h; x_2)} \frac{w_{even}^+(x_4, h; \widetilde{x}_3)}{w_{even}^+(x_3, h; x_2)} = 1 + \mathcal{O}(h)$$

as  $h \rightarrow 0$ . □

Here we return to equation (1.1). The spectral parameter  $\lambda$  near  $\lambda_0$  is an eigenvalue of  $L$  if and only if  $\mathbf{u}_1$  and  $\mathbf{u}_4$  are linearly dependent, since  $\mathbf{u}_1 \in L^2(\mathbb{R}_-)$  and  $\mathbf{u}_4 \in L^2(\mathbb{R}_+)$ . That is, we consider the condition

$$(3.8) \quad \mathcal{W}(\mathbf{u}_1, \mathbf{u}_4) = 0.$$



From (3.2) and (3.3), we know that the Wronskian  $\mathcal{W}(\mathbf{u}_1, \mathbf{u}_4)$  is expressed in terms of  $\mathbf{u}_j$  and  $\tilde{\mathbf{u}}_k$  ( $j, k \in \{2, 3\}$ ) as

$$\mathcal{W}(\mathbf{u}_1, \mathbf{u}_4) = c_2 \tilde{c}_3 \mathcal{W}(\mathbf{u}_2, \tilde{\mathbf{u}}_3) - \tilde{c}_2 c_3 \mathcal{W}(\tilde{\mathbf{u}}_2, \mathbf{u}_3).$$

Since  $\mathbf{u}_j$  and  $\tilde{\mathbf{u}}_j$  are linearly dependent and satisfy

$$\tilde{\mathbf{u}}_2 = e^{-iI(\lambda)/h} \mathbf{u}_2, \quad \tilde{\mathbf{u}}_3 = e^{iI(\lambda)/h} \mathbf{u}_3,$$

condition (3.8) is equivalent to

$$c_2 \tilde{c}_3 e^{iI(\lambda)/h} - \tilde{c}_2 c_3 e^{-iI(\lambda)/h} = 0.$$

That is,  $I(\lambda)$  satisfies

$$I(\lambda) + \frac{h}{2i} \log \left( -\frac{c_2 \tilde{c}_3}{\tilde{c}_2 c_3} \right) = \left( k + \frac{1}{2} \right) \pi h$$

for some integer  $k$ .

From Lemma 3.1, when  $A(\alpha)A(\beta) > 0$ ,

$$\log \left( -\frac{c_2 \tilde{c}_3}{\tilde{c}_2 c_3} \right) = \log(1 + \mathcal{O}(h)) = \mathcal{O}(h) \quad (h \rightarrow 0).$$

On the other hand, when  $A(\alpha)A(\beta) < 0$ ,

$$\log \left( -\frac{c_2 \tilde{c}_3}{\tilde{c}_2 c_3} \right) = \log(-1 + \mathcal{O}(h)) = \pi i + \mathcal{O}(h) \quad (h \rightarrow 0).$$

In conclusion, the quantization condition for eigenvalues  $\lambda$  is given by

$$I(\lambda) = \left( k + \frac{1}{2} \right) \pi h + \mathcal{O}(h^2) \quad (h \rightarrow 0),$$

in the case  $A(\alpha)A(\beta) > 0$ . Similarly, we obtain

$$I(\lambda) = k\pi h + \mathcal{O}(h^2) \quad (h \rightarrow 0),$$

in the case  $A(\alpha)A(\beta) < 0$ .

#### 4. EIGENVALUE PROBLEM FOR THE NON-SELF-ADJOINT CASE

In this section, we consider the eigenvalue problem:

$$(4.1) \quad L_\varepsilon \mathbf{u}(x) = \lambda \mathbf{u}(x), \quad L_\varepsilon := \begin{pmatrix} ih \frac{d}{dx} & -iA_\varepsilon(x) \\ iA_\varepsilon(x) & -ih \frac{d}{dx} \end{pmatrix},$$

for  $A_\varepsilon(x) = A(x) + i\varepsilon B(x)$  with  $\varepsilon > 0$ . First we consider the quantization condition for the eigenvalues. Here we assume that  $A(x)$  satisfies Assumption (A1) and  $B(x)$  is real-valued, analytic and bounded on  $\mathbb{R}$ .

Let  $D(\lambda_0, \varepsilon_0) = \{x \in \mathbb{C}; |x - \lambda_0| < \varepsilon_0\}$  for a positive  $\varepsilon_0$ . Under Assumption (A1), for all  $\lambda \in D(\lambda_0, \varepsilon_0)$  and  $\varepsilon \in (0, \varepsilon_0]$ , there exist zeros of  $A_\varepsilon(x)^2 - \lambda^2$ ,  $\alpha(\lambda, \varepsilon)$  and  $\beta(\lambda, \varepsilon)$  such that  $\alpha(\lambda_0, 0) = \alpha_0$  and  $\beta(\lambda_0, 0) = \beta_0$ . We simply write them as  $\alpha_\varepsilon$  and  $\beta_\varepsilon$ , and define the action integral  $I(\lambda, \varepsilon)$ :

$$(4.2) \quad I(\lambda, \varepsilon) = \int_{\alpha_\varepsilon(\lambda)}^{\beta_\varepsilon(\lambda)} \sqrt{A_\varepsilon(t)^2 - \lambda^2} dt.$$

In addition, the exact WKB solutions for (4.1) are given by replacing  $A(x)$  with  $A_\varepsilon(x)$  in (2.1), and we denote those solutions by  $\mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0)$ .

We choose  $\alpha_\varepsilon$  and  $\beta_\varepsilon$  for the base points of the phase function  $z(x, \varepsilon)$ . The Stokes lines which pass through the points  $\alpha_\varepsilon$  and  $\beta_\varepsilon$  are drawn in Figure 4.

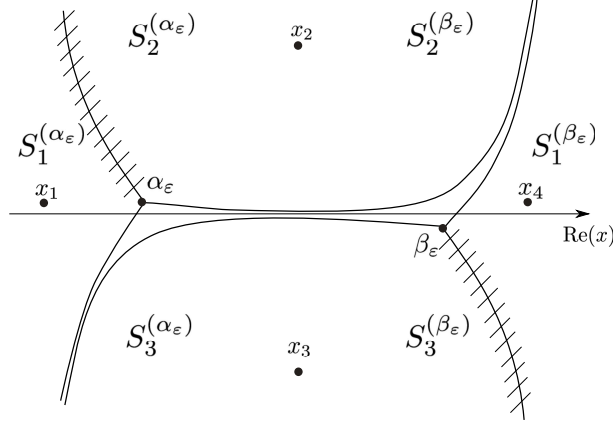


FIGURE 4. The Stokes lines for a sufficiently small  $\varepsilon$ .  $S_j^{(\alpha_\varepsilon)}$  and  $S_k^{(\beta_\varepsilon)}$  indicate the sector which is generated by the Stokes lines emanating from  $\alpha_\varepsilon$  and  $\beta_\varepsilon$  respectively.

The Stokes lines continuously change with respect to  $\varepsilon$  from the case of  $\varepsilon = 0$ , since  $\alpha_\varepsilon$ ,  $\beta_\varepsilon$  and  $z(x, \varepsilon)$  are continuous with respect to  $\varepsilon$ . Here, we assume that  $\varepsilon$  is sufficiently small, and take base points as in Figure 4. Then, we can derive the quantization conditions for eigenvalues of  $L_\varepsilon$  in the same way as the previous section.

**Lemma 4.1.** *Assume (A1), and let  $B(x)$  be real-valued, analytic and bounded on  $\mathbb{R}$ . In the case  $A(\alpha_0) = A(\beta_0)$ , there exist positive constants  $\varepsilon_0$  and  $h_0$ , and a function  $r_+(\lambda, \varepsilon, h)$  bounded on  $D(\lambda_0, \varepsilon_0) \times (0, \varepsilon_0] \times (0, h_0]$  such that  $\lambda \in D(\lambda_0, \varepsilon_0)$  is an eigenvalue of  $L_\varepsilon$  for  $\varepsilon \in (0, \varepsilon_0]$  and  $h \in (0, h_0]$  if and only if*

$$(4.3) \quad I(\lambda, \varepsilon) = \left(k + \frac{1}{2}\right) \pi h + h^2 r_+(\lambda, \varepsilon, h)$$

*holds for some integer  $k$ . In the case  $A(\alpha_0) = -A(\beta_0)$ , there exist positive constants  $\varepsilon_0$  and  $h_0$ , and a function  $r_-(\lambda, \varepsilon, h)$  bounded on  $D(\lambda_0, \varepsilon_0) \times (0, \varepsilon_0] \times (0, h_0]$  such that  $\lambda \in D(\lambda_0, \varepsilon_0)$  is an eigenvalue of  $L_\varepsilon$  for  $\varepsilon \in (0, \varepsilon_0]$  and  $h \in (0, h_0]$  if and only if*

$$(4.4) \quad I(\lambda, \varepsilon) = k\pi h + h^2 r_-(\lambda, \varepsilon, h)$$

*holds for some integer  $k$ .*

Now, we assume Assumption (A2) for  $A_\varepsilon(x)$ , which results in a symmetry of the action integral  $I(\lambda, \varepsilon)$  and the exact WKB solutions  $\mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0)$  with respect to complex conjugation.

**Lemma 4.2.** *Under Assumption (A2), the action integral  $I(\lambda, \varepsilon)$  is equal to the complex conjugate of  $I(\bar{\lambda}, \varepsilon)$ :*

$$\overline{I(\bar{\lambda}, \varepsilon)} = I(\lambda, \varepsilon).$$

*Proof.* By a simple calculation, we find that  $-\bar{\beta}_\varepsilon$  and  $-\bar{\alpha}_\varepsilon$  are zeros of  $A_\varepsilon(x)^2 - \bar{\lambda}^2$  under Assumption (A2). That is,  $I(\bar{\lambda}, \varepsilon)$  is represented as

$$I(\bar{\lambda}, \varepsilon) = \int_{-\bar{\beta}_\varepsilon}^{-\bar{\alpha}_\varepsilon} \sqrt{\bar{\lambda}^2 - A_\varepsilon(t)^2} dt.$$

We take the complex conjugate of this, and obtain that

$$\overline{I(\bar{\lambda}, \varepsilon)} = \int_{-\bar{\beta}_\varepsilon}^{-\bar{\alpha}_\varepsilon} \sqrt{\overline{(\bar{\lambda}^2 - A_\varepsilon(t)^2)}} d\bar{t} = \int_{-\bar{\beta}_\varepsilon}^{-\bar{\alpha}_\varepsilon} \sqrt{\lambda^2 - \overline{A_\varepsilon(t)^2}} d\bar{t}.$$

Then, we change the variable from  $t$  to  $-\bar{t}$ ,

$$\overline{I(\bar{\lambda}, \varepsilon)} = - \int_{\beta_\varepsilon}^{\alpha_\varepsilon} \sqrt{\lambda^2 - \overline{A_\varepsilon(-\bar{t})^2}} dt = \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \sqrt{\lambda^2 - A_\varepsilon(t)^2} dt.$$

This is just the action integral.  $\square$

We denote the exact WKB solutions for the equation

$$(4.5) \quad L_\varepsilon \mathbf{v}(x) = \bar{\lambda} \mathbf{v}(x)$$

by  $\mathbf{v}^\pm$ . Then,  $\mathbf{v}^\pm$  is obtained by replacing  $\lambda$  with  $\bar{\lambda}$ . Under Assumption (A2),  $\mathbf{u}^\pm$  and  $\mathbf{v}^\pm$  also have the following symmetry relations.

**Lemma 4.3.** *Under Assumption (A2), if  $\overline{A_\varepsilon(-\bar{x})} = A_\varepsilon(x)$ , the exact WKB solutions  $\mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0)$  and  $\mathbf{v}^\pm(x, h, \varepsilon; \gamma, x_0)$  satisfy*

$$\mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\mathbf{v}^\mp(-\bar{x}, h, \varepsilon; -\bar{\gamma}, -\bar{x}_0)},$$

and if  $\overline{A_\varepsilon(-\bar{x})} = -A_\varepsilon(x)$ , then

$$\mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{\mathbf{v}^\mp(-\bar{x}, h, \varepsilon; -\bar{\gamma}, -\bar{x}_0)}.$$

*Proof.* Let  $\overline{A_\varepsilon(-\bar{x})} = A_\varepsilon(x)$ . By taking the complex conjugate and changing the variable  $x$  to  $-\bar{x}$  for the functions  $z$  and  $\mathbf{w}^\pm$  of the solutions  $\mathbf{v}^\pm$ , we obtain

$$\overline{z(-\bar{x}, \varepsilon; -\bar{\gamma}; \bar{\lambda})} = -z(x, \varepsilon; \gamma; \lambda),$$

and

$$\overline{\mathbf{w}^\pm(-\bar{x}, h, \varepsilon; -\bar{x}_0; \bar{\lambda})} = \mathbf{w}^\mp(x, h, \varepsilon; x_0; \lambda).$$

In the same way, for the matrix function  $Q(x, \varepsilon)$  of  $\mathbf{v}^\pm$ ,

$$\overline{Q(-\bar{x}, \varepsilon; \bar{\lambda})} = Q(x, \varepsilon; \lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here, we recall that the solutions  $\mathbf{v}^\pm$  is of the form

$$\mathbf{v}^\pm(x, h, \varepsilon; \gamma, x_0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} e^{\pm z(x, \varepsilon; \gamma)/h} Q(x, \varepsilon) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\frac{1 \pm 1}{2}} \mathbf{w}^\pm(x, h, \varepsilon; x_0).$$

By taking the complex conjugate and changing the variable  $x$  to  $-\bar{x}$ , and using above, we obtain the first relation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\mathbf{v}^\mp(-\bar{x}, h, \varepsilon; -\bar{\gamma}, -\bar{x}_0)} = \mathbf{u}^\pm(x, h, \varepsilon; \gamma, x_0).$$

If  $\overline{A_\varepsilon(-\bar{x})} = -A_\varepsilon(x)$ , then we find that

$$\begin{aligned} \overline{z(-\bar{x}, \varepsilon; -\bar{\gamma}; \bar{\lambda})} &= -z(x, \varepsilon; \gamma; \lambda), \\ \overline{\mathbf{w}^\pm(-\bar{x}, h, \varepsilon; -\bar{x}_0; \bar{\lambda})} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{w}^\mp(x, h, \varepsilon; x_0; \lambda), \end{aligned}$$

and

$$\overline{Q(-\bar{x}, \varepsilon; \bar{\lambda})} = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q(x, \varepsilon; \lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From this property, the second relation also follows.  $\square$

Here we take the base points  $x_1 \in S_1^{(\alpha_\varepsilon)}$  and  $x_2 \in S_2^{(\alpha_\varepsilon)}$  so that  $-\bar{x}_1 \in S_1^{(\beta_\varepsilon)}$  and  $-\bar{x}_2 \in S_3^{(\beta_\varepsilon)}$ , and set the exact WKB solutions for (4.1):

$$\begin{cases} \mathbf{u}_1 = \mathbf{u}^+(x, h, \varepsilon; \alpha_\varepsilon, x_1), & \mathbf{u}_2 = \mathbf{u}^+(x, h, \varepsilon; \alpha_\varepsilon, x_2), \\ \mathbf{u}_3 = \mathbf{u}^-(x, h, \varepsilon; \alpha_\varepsilon, -\bar{x}_2) & \mathbf{u}_4 = \mathbf{u}^-(x, h, \varepsilon; \beta_\varepsilon, -\bar{x}_1). \end{cases}$$

In addition, let us define a function  $W(\lambda, \varepsilon)$  by the Wronskian of  $\mathbf{u}_1$  and  $\mathbf{u}_4$ , that is,

$$W(\lambda, \varepsilon) := \mathcal{W}(\mathbf{u}_1, \mathbf{u}_4).$$

We also take the solutions for (4.5) as

$$\begin{cases} \mathbf{v}_1 = \mathbf{v}^+(x, h, \varepsilon; -\bar{\beta}_\varepsilon, x_1), & \mathbf{v}_2 = \mathbf{v}^+(x, h, \varepsilon; -\bar{\alpha}_\varepsilon, x_2), \\ \mathbf{v}_3 = \mathbf{v}^-(x, h, \varepsilon; -\bar{\alpha}_\varepsilon, -\bar{x}_2) & \mathbf{v}_4 = \mathbf{v}^-(x, h, \varepsilon; -\bar{\alpha}_\varepsilon, -\bar{x}_1). \end{cases}$$

Then, we see that

$$(4.6) \quad W(\lambda, \varepsilon) = \pm \overline{W(\bar{\lambda}, \varepsilon)},$$

by the definition of  $W(\lambda, \varepsilon)$  and applying Lemma 4.3. Here, the sign of (4.6) is dependent on whether  $A_\varepsilon(x) = \overline{A(-\bar{x})}$  or  $A_\varepsilon(x) = -\overline{A(-\bar{x})}$ .

Now, we recall that  $W(\lambda, \varepsilon)$  is represented as

$$(4.7) \quad W(\lambda, \varepsilon) = a(\lambda, \varepsilon, h) e^{iI(\lambda, \varepsilon)/h} + b(\lambda, \varepsilon, h) e^{-iI(\lambda, \varepsilon)/h},$$

where  $a$  and  $b$  are some functions with  $a = 1 + \mathcal{O}(h)$  and  $b = 1 + \mathcal{O}(h)$  or  $-1 + \mathcal{O}(h)$  as  $h \rightarrow 0$ . In particular,  $a$  and  $b$  satisfy

$$(4.8) \quad a(\lambda, \varepsilon, h) = \pm \overline{b(\bar{\lambda}, \varepsilon, h)},$$

since  $W(\lambda, \varepsilon)$  satisfy (4.6), and  $I(\lambda, \varepsilon)$  also satisfy Lemma 4.2. Moreover, (4.7) is rewritten as

$$W(\lambda, \varepsilon) = b(\lambda, \varepsilon, h) e^{-I(\lambda, \varepsilon)/h} \left( \exp \left( \frac{2i}{h} (I(\lambda, \varepsilon) + h^2 r(\lambda, \varepsilon, h)) \right) + 1 \right),$$

where

$$r(\lambda, \varepsilon, h) = \frac{1}{2ih} \log \frac{a(\lambda, \varepsilon, h)}{b(\lambda, \varepsilon, h)}.$$

Then, we use (4.8) and obtain that

$$(4.9) \quad r(\lambda, \varepsilon, h) = \overline{r(\bar{\lambda}, \varepsilon, h)}.$$

Here we take  $I(\lambda, \varepsilon, h)$  as

$$I(\lambda, \varepsilon, h) := I(\lambda, \varepsilon) + h^2 r(\lambda, \varepsilon, h).$$

This is a function from a neighborhood of  $\lambda_0$  to one of  $I(\lambda_0, 0)$ . In particular,  $r(\lambda, \varepsilon, h)$  is holomorphic near  $\lambda_0$ , and  $I(\lambda, \varepsilon)$  satisfies  $\frac{dI}{d\lambda}(\lambda_0, 0) \neq 0$ . This implies that  $I(\lambda, \varepsilon, h)$  has an inverse function  $I^{-1}(\zeta, \varepsilon, h)$ , and the eigenvalues of  $L_\varepsilon$  near  $\lambda_0$  are given by

$$\lambda_{\varepsilon, k} = I^{-1}(c_k \pi h, \varepsilon, h), \quad k \in \mathbb{Z},$$

where  $c_k = k$  or  $k + 1/2$ . In addition, we know that  $I(\lambda, \varepsilon, h)$  is real for  $\lambda \in \mathbb{R}$ , since  $r(\lambda, \varepsilon, h)$  and  $I(\lambda, \varepsilon)$  is real for  $\lambda \in \mathbb{R}$  by (4.9) and Lemma 4.2. That is, the eigenvalues near  $\lambda_0$  are real.

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